Hamiltonicity of mildly pseudorandom regular graphs

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Abstract

We show that if an (n, d, λ)-graph satisfies $\lambda \leq (1-\delta)d$ and $d \gg \delta^{-6}(\log n)^3$ for some $\delta > 0$, then it is Hamiltonian. A qualitatively similar result was recently proven by Bradač and Janzer. Our proof here is shorter and gives better quantitative bounds.

In our proof, as in earlier work of Ferber, Han, Mao, and Vershynin, we use a random matrix inequality to show that a mild spectral gap is typically preserved after randomly sampling an appropriate proportion of the vertices. This allows us to deduce that typical balanced bipartite subgraphs of pseudorandom graphs contain perfect matchings. To convert a collection of perfect matchings into a Hamilton cycle, we use a variant of the sorting network method.

1 Introduction

An (n, d, λ)-graph is an n-vertex d-regular graph where all non-trivial eigenvalues of the adjacency matrix have absolute value at most λ. Roughly speaking, the smaller λ is compared to d, the more pseudorandom the underlying graph is and the better it expands. For example, it is well-known that (n, d, λ)-graphs have their edge expansion controlled by $d - \lambda$, see for example [15, Theorem 2.4].

An important milestone in the study of Hamiltonicity is the recent resolution of the Krivelevich–Sudakov conjecture [17] by Draganić, Montgomery, Correia, Pokrovskiy, and Sudakov [10]. This conjecture, now a theorem, asserts that (n, d, λ)-graphs are Hamiltonian when $\lambda \leq d/C$, where C is a sufficiently large constant. Although this is a widely useful criterion for Hamiltonicity, and covers many interesting cases such as random Cayley graphs with $C \log n$ generators, it is not powerful enough to deduce Hamiltonicity of various other graph families, such as Kneser graphs [20], or recover the classical fact that hypercubes contain Hamilton cycles.

The motivation of the present paper is to investigate to what extent the Krivelevich–Sudakov conjecture can be strengthened. In particular, we would like to determine the weakest possible assumption on an (n, d, λ)-graph that forces it to be Hamiltonian. Of course, any Hamiltonian graph has a near perfect matching (meaning a perfect matching that misses at most one vertex) and is connected. Krivelevich and Sudakov showed that $\lambda \leq d - 2$ is sufficient to guarantee near perfect matchings [18, Theorem 4.3] and it is a standard exercise to check that $\lambda < d$ ensures connectivity. These two properties by themselves are not enough to establish Hamiltonicity; however, a widespread phenomenon in graph theory is that any non-trivial condition that implies that a graph has a near perfect matching and is connected also implies that the graph is Hamiltonian.

The preceding discussion raises the natural question of whether all (n, d, λ)-graphs with $\lambda \leq d - 2$ are Hamiltonian. The author is not aware of any counterexamples to such a conjecture, despite extensive search with modern AI tools. This should of course be taken with a grain of salt, as it remains very difficult to argue, in general, that a given graph is not Hamiltonian.

A more plausible conjecture is that a wider separation between d and λ should be sufficient for Hamiltonicity.

Conjecture 1.1 (Folklore)**.** *For any $\delta \in (0, 1)$, there exists d_0 so that for every $d \geq d_0$, every (n, d, λ)-graph with $\lambda \leq (1-\delta)d$ is Hamiltonian.*

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The author heard the above conjecture formulated by Ferber and stronger forms of it were given in print by Bradač and Janzer [5] and by Montgomery [21]. The author also heard the conjecture formulated as a question in various seminars, for example by Bucić and by Gowers.

Bradač and Janzer made significant progress towards Conjecture 1.1, essentially resolving a strong version of it whenever the host graph has polylogarithmic degree. In particular, they showed the following.

Theorem 1.2 (Bradač–Janzer)**.** *Let G be an (n, d, λ)-graph, and suppose that n is sufficiently large. If $\lambda \leq (1-\delta)d$ and $d \geq (\delta^{-1}\log n)^{10^9}$, then G is Hamiltonian.*

Our contribution in this paper is a shorter proof of the above that comes with better quantitative estimates.

Theorem 1.3. *Let G be an (n, d, λ)-graph. If $\lambda \leq (1-\delta)d$ and $d \geq C \cdot \delta^{-6}(\log n)^3$, where $\delta \in (0,1)$ and C is a sufficiently large absolute constant, then G is Hamiltonian.*

We also record a combinatorial version of the above result that can be obtained by applying Cheeger's inequality (see, for example, [5, Lemmas 3.3 and 3.6]). We say that a graph G is ε-far from bipartite if at least $\varepsilon e(G)$ edges must be deleted from G in order to make it bipartite.

Corollary 1.4. *There is an absolute constant $C > 0$ such that the following holds. Let G be an n-vertex d-regular γ-expander, in the sense that $e_G(S, V(G) \setminus S) \geq \gamma d|S|$ for every $S \subseteq V(G)$ with $1 \leq |S| \leq 2n/3$. Suppose moreover that G is ε-far from bipartite. If n is sufficiently large and $d \geq C\varepsilon^{-12}\gamma^{-12}(\log n)^3$, then G is Hamiltonian.*

An analogous result can be obtained for bipartite graphs with one-sided spectral gap as well, as was done in [5], but we do not pursue this here.

Remarks on the proof, acknowledgements, and AI declaration. An important ingredient in the proof is Lemma 2.2, which asserts, in a suitable weak sense, that random vertex subgraphs of (n, d, λ)-graphs retain pseudorandomness. Its proof uses only standard matrix concentration estimates, and the underlying phenomenon is not new. A substantially more sophisticated version was proved by Chen, Khanna, and Li [9], and by Assadi, Kapralov, and Yu [3], although with worse quantitative bounds.

The author is also indebted to the work of Ferber, Han, Mao, and Vershynin [11], which first introduced him to these ideas. Their work proves a related sampling statement that yields partial progress towards the Krivelevich–Sudakov conjecture. The argument there has a constant factor loss as an artifact of the proof, which prevents a direct application to the mildly pseudorandom graphs considered in the present paper. As the author learned from Ferber, an unpublished variant of their argument avoids this loss at the expense of obtaining only a one-sided spectral estimate.

AI tools were used to help identify a short proof of the particular one-sided statement formulated here as Lemma 2.2. The underlying result is nevertheless attributed to Ferber, Han, Mao, and Vershynin. The formulation used in the present paper is weaker than what their methods appear capable of proving, but has the advantage of admitting a particularly short proof, consisting essentially of two applications of the matrix Bernstein inequality.

The remaining ingredients of the proof build on the work of other researchers. In particular, we build on the sorting network method, which goes back to work of Häggkvist and Thomasson [12, 13]. We use a version of this method inspired by our prior work with Hyde, Morrison, and Pavez–Signé [16], and in particular, we use a simplification later discovered in our work with Bowtell, Montgomery, and Pokrovskiy [4] that uses Hamilton routers. Other important ingredients in the proof include Hall's marriage theorem and a very useful hypergraph generalisation thereof, due to Haxell [14], which we use to find connections between prescribed pairs of vertices in expander graphs, as is standard in the area. The contribution of the present paper is assembling these tools in a convenient fashion.

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2 Spectral results, subsampling expanders

In this section, we record two important spectral facts. The first, Lemma 2.1, is quite standard, and says that spectral gap implies a combinatorial expansion condition. The second, Lemma 2.2, states that randomly sampling rows and columns of a biadjacency matrix does not change the operator norm significantly, and follows by basic matrix inequalities, but appears to be not so well-known in extremal combinatorics. As stated in the acknowledgements, versions of such a statement were proved before in [3, 11, 9]. The proof we use here, a routine application of the Matrix Bernstein inequality, was suggested to us by an AI model.

We first record the following standard consequence of [6, Corollary 4.7.4 and Theorem 2.8.1(iii)] which allows us to translate spectral bounds into a combinatorial expansion property. We will not need it until the proof of Theorem 1.3.

Definition. A graph G is *p-cut-dense* if $e_G(S, V(G) \setminus S) \geq p|S||V(G) \setminus S|$ for every $S \subseteq V(G)$.

Lemma 2.1. *Let G be a graph on $n \geq 2$ vertices, and let $\lambda_2(G)$ denote the second largest eigenvalue of its adjacency matrix. If $\delta(G) > \lambda_2(G)$, then G is $\frac{\delta(G)-\lambda_2(G)}{n}$-cut-dense.*

Given the previous lemma, to retain combinatorial expansion after random subsampling in a (n, d, λ)-graph, we need a way to deduce information about the spectrum of the randomly sampled subgraph. The following is the lemma that allows us to achieve this. The statement is essentially due to Ferber, Han, Mao, and Vershynin, as explained in the introduction, however, here, we give a short proof, discovered with AI assistance.

Lemma 2.2 (Expansion inheritance under vertex subsampling)**.** *For every $a > 0$, the constant $C_a = 30000(a + 4)$ has the following property. Let G be an (n, d, λ)-graph with adjacency matrix A, and suppose that $\lambda \leq (1 - \delta)d$. Let $m \geq 2$ be an integer, set $p = m/n$ and $D = pd$, and assume that $p \leq \frac{\delta}{100}$ and $D \geq C_a\delta^{-2} \log n$. Choose (U, V) uniformly among all ordered pairs of disjoint m-subsets of $V(G)$, and let $K = A[U, V]$ be the biadjacency matrix of $G[U, V]$. With probability at least $1 - n^{-a}$, the following holds.*

$$s_2(K) \leq \left(1 - \frac{\delta}{2}\right) D. \tag{1}$$

To prove the above, we use the following form of matrix Bernstein inequality. Here, $\lambda_{\max}(\cdot)$ returns the algebraically largest eigenvalue of a self-adjoint matrix and X_j^2 is just matrix multiplication. For a real matrix F, $\|F\|$ denotes its operator norm with respect to the Euclidean norm.

Theorem 2.3 (Matrix Bernstein [22, Theorem 1.4])**.** *Let X_k be a finite sequence of independent random self-adjoint matrices of dimension d. Suppose that $\mathbb{E}X_k = 0$ and $\lambda_{\max}(X_k) \leq R$ almost surely for every k. Then, for every $t \geq 0$,*

$$\mathbb{P}\left\{\lambda_{\max}\left(\sum_k X_k\right) \geq t\right\} \leq d \exp\left(-\frac{t^2/2}{\sigma^2 + Rt/3}\right),$$

where

$$\sigma^2 := \left\|\sum_k \mathbb{E}(X_k^2)\right\|.$$

In particular, for every $t > 0$, with probability at least $1 - re^{-t}$,

$$\lambda_{\max}\left(\sum_{j=1}^{s} X_j\right) \leq \sqrt{2\sigma^2 t} + \frac{2}{3}Rt.$$

For a real matrix F, let $s_1(F) \geq s_2(F) \geq \cdots$ denote its singular values, i.e., the square roots of the eigenvalues of $F^{\mathsf{T}}F$ in nonincreasing order. Thus $\|F\| = s_1(F)$. We write $F[I, J]$ for the submatrix with row set I and column set J, and $J_{r,s}$ for the $r \times s$ all-ones matrix; when $r = s$, we abbreviate this to J_r. For a real matrix F, we write F^{T} for its transpose. For self-adjoint matrices A and B, we write $A \preceq B$ if

$B - A$ is positive semidefinite; equivalently, $x^\mathsf{T} A x \le x^\mathsf{T} B x$ for every vector x. In particular, if $0 \preceq A \preceq B$, then $\|A\| \le \|B\|$. For a vector $f \in \mathbb{R}^r$, the matrix ff^T is the rank-one positive semidefinite matrix whose (i,j)-entry is $f_i f_j$.

The proof of Lemma 2.2 follows by applying the following lemma twice, once to sample the rows, and once to sample the columns.

Lemma 2.4. *Let $F \in \mathbb{R}^{r\times s}$, where $r, s \ge 1$, with columns $f_1, \dots, f_s$, and set $c = \max_{j\in[s]} \|f_j\|_2$. If I is a uniformly random k-subset of $[s]$, where $0 \le k \le s$, and $q = k/s$, let F_I be the submatrix consisting of the columns indexed by I. Then, for every $t > 0$,*

$$\mathbb{P}\Big(\|F_I\| > \sqrt{q}\,\|F\| + c\sqrt{t}\Big) \le r(s+1)e^{-t}. \tag{2}$$

Proof. We assume that $0 < k < s$ and $c > 0$, as otherwise the statement is trivial. First select columns independently with probability q. Write $f_j \in \mathbb{R}^r$ for column j of F, and let ξ_j be its indicator random variable. If F_I denotes the selected matrix, then we observe that

$$F_I F_I^\mathsf{T} = \sum_{j=1}^{s} \xi_j f_j f_j^\mathsf{T}. \tag{3}$$

Note also that $\|F_I\|^2 = \lambda_{\max}(F_I F_I^\mathsf{T})$. To apply Matrix Bernstein, we need to center the matrices, so set $X_j = (\xi_j - q) f_j f_j^\mathsf{T}$, noting $\mathbb{E}X_j = 0$. Since $\|f_j f_j^\mathsf{T}\| = \|f_j\|_2^2$, we have $\|X_j\| \le c^2$. Moreover,

$$\begin{aligned}\sum_j \mathbb{E}(X_j^2) &= q(1-q)\sum_j \|f_j\|_2^2\, f_j f_j^\mathsf{T} \\ &\preceq qc^2 \sum_j f_j f_j^\mathsf{T} = qc^2 F F^\mathsf{T}.\end{aligned}$$

The variance parameter in Theorem 2.3 is therefore at most $qc^2\|F\|^2$. Theorem 2.3 implies that (using the 'in particular' part), with failure probability at most re^{-t}, we have

$$\begin{aligned}\|F_I\|^2 &\le q\|F\|^2 + \sqrt{2q}\,c\,\|F\|\sqrt{t} + \frac{2}{3}c^2 t \\ &\le \big(\sqrt{q}\,\|F\| + c\sqrt{t}\big)^2.\end{aligned}$$

This proves the desired bound under binomial sampling.

To pass to a uniform k-set, condition on $\sum_j \xi_j = k$. For $q = k/s$, the integer k is a mode of $\mathrm{Bin}(s,q)$, so

$$\mathbb{P}\left(\sum_j \xi_j = k\right) \ge \frac{1}{s+1}. \tag{4}$$

Consequently, conditioning increases the failure probability by at most $s+1$, which proves (2). □

We can now start the proof of the subsampling lemma.

Proof of Lemma 2.2. We denote $V(G) = [n]$ and set $t = (a+4)\log n$, $\eta = \frac{\delta}{100}$. Our choice of C_a gives $D \ge 30000\delta^{-2}t$.

We first define a centered version of the matrix as $B = A - \frac{d}{n}J_n$, noting $\|B\| \le (1-\delta)d$. Observe, by d-regularity, every row and column of B has squared Euclidean norm

$$d\left(1 - \frac{d}{n}\right)^2 + (n-d)\left(\frac{d}{n}\right)^2 = d - \frac{d^2}{n} \le d. \tag{5}$$

Also $K = (d/n)J_m + B[U,V]$, so we can deduce

$$s_2(K) \le \|B[U,V]\|. \tag{6}$$

Above we used the inequality $s_2(M) \le \|M - R\|$ that holds when $\operatorname{rank} R \le 1$, which follows from the min-max characterization of s_2.

We first sample the rows. By transposing, we may apply Lemma 2.4 to the rows, and with (5), we obtain

$$\|B[U,[n]]\| \le \sqrt{p}\,\|B\| + \sqrt{dt} \tag{7}$$

except with probability at most $n(n+1)e^{-t}$. By Chernoff's bound and a union bound, we also have, with very high probability, for every $j \in [n]$,

$$\begin{aligned}\|B[U,\{j\}]\|_2^2 &= \sum_{i\in U}\left(A_{ij} - \frac{d}{n}\right)^2 \\ &\le d_G(j,U) + m\left(\frac{d}{n}\right)^2 \le 2D + \frac{d}{n}D \le 3D.\end{aligned} \tag{8}$$

We now condition on (7) and (8), and sample the columns. The set V is a uniform m-subset of $[n]\setminus U$, so its proportion is $q = \frac{m}{n-m} = \frac{p}{1-p}$. Apply Lemma 2.4 to the matrix $F = B[U,[n]\setminus U]$. With failure probability at most $m(n-m+1)e^{-t} \le n(n+1)e^{-t}$, we have that

$$\|B[U,V]\| \le \sqrt{q}\,\|F\| + \max_{j\notin U}\|B[U,\{j\}]\|_2\,\sqrt{t}. \tag{9}$$

Since this conditional failure bound is independent of the choice of U, it also holds unconditionally after averaging over U.

Deleting columns cannot increase the operator norm, so $\|F\| \le \|B[U,[n]]\|$. Conditional on the high probability events, (7), (8), and (9), we have that

$$\begin{aligned}\|B[U,V]\| &\le \sqrt{\frac{p}{1-p}}\big(\sqrt{p}\,\|B\| + \sqrt{dt}\big) + \sqrt{3Dt} \\ &= \frac{p}{\sqrt{1-p}}\,\|B\| + \left(\frac{1}{\sqrt{1-p}} + \sqrt{3}\right)\sqrt{Dt} \\ &\le \frac{p}{\sqrt{1-p}}\,\|B\| + 4\sqrt{Dt},\end{aligned} \tag{10}$$

where in the last inequality we used that $p \le 1/2$.

We now combine everything. By (6) and using $D \ge 30000\delta^{-2}t$, we have $\frac{s_2(K)}{D} \le \frac{1-\delta}{\sqrt{1-p}} + 4\sqrt{\frac{t}{D}}$. Using $p \le \delta/100$, $(1-p)^{-1/2} \le 1+p$, and (8), we obtain

$$\frac{s_2(K)}{D} \le 1 - \delta + p + \frac{\delta}{4} \le 1 - \frac{\delta}{2}.$$

This proves the desired spectral bound (1) conditional on the good events. The sum of the failure probabilities in (8) and the two matrix sampling steps is at most $6n(n+1)e^{-t} = 6n(n+1)n^{-(a+4)} \le n^{-a}$, as desired. □

3 Making connections

The following lemma allows us to connect a collection of pairs of vertices with vertex-disjoint paths in mildly expanding graphs. We will use it in the next section to build a Hamilton router.

Lemma 3.1 (Connecting lemma)**.** *There is an absolute constant $C > 0$ such that the following holds. Let G be a graph and let $R \subseteq V(G)$ with $|R| \ge 3$. Suppose that $G[R]$ is p-cut-dense, write $\Delta := \Delta(G[R])$, and assume that $p|R| \ge \gamma\Delta$ for some $0 < \gamma \le 1$. Let $(a_i,b_i)_{i=1}^k$ be pairs of vertices in $V(G)\setminus R$, and let $T := \{a_i,b_i : i \in [k]\}$. Let $\beta \in (0,1)$ and suppose that every vertex occurs at most 10 times among $a_1,b_1,\dots,a_k,b_k$, that $d_G(x,R) \ge \beta\Delta$ for every $x \in T$, and that $d_G(v,T) \le \frac{\beta\gamma^2\Delta}{C\log(2|R|)}$ for every $v \in R$.*

Then, there exist a_i–b_i paths P_i, $i \in [k]$, whose internal vertex sets are pairwise disjoint and contained in R, and such that $|P_i| \le \frac{C}{\gamma}\log(2|R|)$ for every $i \in [k]$.

One can compare the statement with [8, Definition 3.3 and Lemma 3.4] and [19, Lemma 5.1], which obtains a similar conclusion with an additional property that the paths found are contained within a random reservoir. In our lemma, the reservoir R itself is assumed to expand well, hence the reachability arguments in [8, 19] are not necessary. In our application, R will still be a random subset, and we will show that R inherits a cut-density condition from the expander it is sampled from, using the result in the previous section, Lemma 2.2.

The proof thus reduces to a standard application of Haxell's hypergraph matching theorem [14], which we state below, as it was stated as Theorem 2 in [2]. Recall that $\tau(\mathcal{H})$ is the minimum number of vertices of B needed to meet every edge of a hypergraph $\mathcal{H}$.

Theorem 3.2 (Haxell)**.** *Let $\mathcal{H}$ be a bipartite hypergraph with vertex partition $A \cup B$, where every edge $e \in E(\mathcal{H})$ satisfies $|e \cap A| = 1$ and $|e \cap B| \leq r-1$. For $S \subseteq A$, let $\mathcal{H}_S$ be the hypergraph on B whose edge set is $E(\mathcal{H}_S) := \{e \cap B : e \in E(\mathcal{H}),\ e \cap S \neq \varnothing\}$. If $\tau(\mathcal{H}_S) > (2r-3)(|S|-1)$ for every $S \subseteq A$, then $\mathcal{H}$ has a matching saturating A.*

We essentially model our argument based on Section 4.1 in [7], the main difference is that [7] is concerned with edge-disjointness, as opposed to vertex-disjointness.

Proof of Lemma 3.1. Set $\ell := C_0\gamma^{-1}\log(2|R|)$ with C_0 a sufficiently large constant. Thus, ℓ denotes the target upper bound on the length of the connecting paths we wish to find. We will later choose $C \gg C_0$. We begin with showing that after deleting a few vertices, there still remains a suitable pair $a_i - b_i$ with a short connection, avoiding the deleted vertices.

Claim 1. For every $I \subseteq [k]$ and $U \subseteq R$ with $|U| \leq (2\ell-3)(|I|-1)$, there is some $i \in I$ and a $a_i - b_i$ path of length at most ℓ whose internal vertices are in $R \setminus U$.

We first show how to conclude the proof. Define an auxiliary bipartite hypergraph $\mathcal{H}$ on vertex-partition $A \cup B$, where A corresponds to the k pairs $a_i - b_i$, and B corresponds to vertices of R, and put an edge $a \cup b$ where $a \in A$ and $b \subseteq B$ if the vertices of b form the internal vertices of a $a_i - b_i$ path of length at most ℓ. Note that a matching saturating A is exactly what we are looking for. Furthermore, the assumption of the claim translates exactly to the vertex-cover hypothesis of Haxell's theorem, as desired. It remains to check the claim.

Proof of claim. Write $n := |R|$ and $H := G[R]$. We assume that C_0 is a sufficiently large absolute constant and that C is sufficiently large compared with C_0. Recall that $\ell = C_0\gamma^{-1}\log(2n)$.

We first give an upper bound on the size of I. Using that every vertex occurs at most 10 times among the endpoints of the pairs, we have $2\beta\Delta|I| \leq \sum_{i\in I}\big(d_G(a_i,R) + d_G(b_i,R)\big) \leq 10\sum_{v\in R} d_G(v,T) \leq \frac{10\beta\gamma^2\Delta n}{C\log(2n)}$. Hence $|I| \leq \frac{5\gamma^2 n}{C\log(2n)}$. It follows from the assumption on U that $|U| \leq 2\ell|I| \leq \frac{10C_0}{C}\gamma n$. In particular, taking C sufficiently large compared with C_0, we may assume that $|U| \leq \gamma n/100$.

We next find a large expanding subgraph of $H - U$. Let $W \subseteq R \setminus U$ be a set of maximum size, subject to $|W| \leq 3n/4$ and $e_H\big(W, R \setminus (U \cup W)\big) < \frac{\gamma\Delta}{16}|W|$, taking $W = \varnothing$ if there is no non-empty set with this property. We claim that $|W| \leq \frac{6|U|}{\gamma}$. Indeed, if $W \neq \varnothing$, then cut-density, $|W| \leq 3n/4$, and $pn \geq \gamma\Delta$ give $e_H(W, R\setminus W) \geq p|W|(n-|W|) \geq \frac{\gamma\Delta}{4}|W|$. Consequently, $e_H(W,U) > \frac{3\gamma\Delta}{16}|W|$. Since $e_H(W,U) \leq \Delta|U|$, the claimed bound follows.

Set $K := H - (U \cup W)$. We claim that every $S \subseteq V(K)$ with $|S| \leq |V(K)|/2$ satisfies $e_K(S, V(K)\setminus S) \geq \frac{\gamma\Delta}{16}|S|$. Suppose otherwise, and put $W' := W \cup S$. By the bound on W and the fact that $|U| \leq \gamma n/100$, we have $|W'| \leq |W| + \frac{n-|U|-|W|}{2} < \frac{3n}{4}$. Moreover, $e_H\big(W', R\setminus(U\cup W')\big) \leq e_H\big(W, R\setminus(U\cup W)\big) + e_K(S, V(K)\setminus S) < \frac{\gamma\Delta}{16}|W'|$, contradicting the maximality of W.

As $\Delta(K) \leq \Delta$, it follows that $|N_K(S) \setminus S| \geq \frac{\gamma}{16}|S|$ whenever $|S| \leq |V(K)|/2$. Thus K is a $\gamma/16$-vertex-expander. By a standard ball growth argument, K has diameter at most $64\gamma^{-1}\log(2n)$. Taking C_0 sufficiently large, we may therefore assume that $\operatorname{diam}(K) + 2 \leq \ell$.

It remains to show that some pair has both endpoints adjacent to K. Suppose not. For every $i \in I$, choose $x_i \in a_i, b_i$ with $N_G(x_i, R) \subseteq U \cup W$. Since each vertex occurs at most 10 times among the endpoints, $\beta\Delta|I| \leq \sum_{i\in I} d_G(x_i, U\cup W) \leq 10\sum_{v\in U\cup W} d_G(v,T)$. Using $|W| \leq 6|U|/\gamma$, the hypothesis on $d_G(v,T)$,

and $\gamma \leq 1$, the right hand side is at most $10\left(|U| + \frac{6|U|}{\gamma}\right)\frac{\beta\gamma^2\Delta}{C\log(2n)} \leq \frac{70\beta\gamma\Delta|U|}{C\log(2n)} \leq \frac{140C_0}{C}\beta\Delta|I|$. This is a contradiction once C is chosen sufficiently large compared with C_0.

Hence, for some $i \in I$, both a_i and b_i have a neighbour in K. Let $x \in N_G(a_i) \cap V(K)$ and $y \in N_G(b_i) \cap V(K)$. A shortest x–y path in K, together with the edges a_ix and yb_i, gives an a_i–b_i path whose internal vertices lie in $R \setminus U$ and whose length is at most $\mathrm{diam}(K) + 2 \leq \ell$, as required. □

This concludes the proof. □

4 Hamilton routers

We require the definition of a Hamilton router and an explicit construction as given in Definition 4.3. Both are borrowed entirely from our previous work with Bowtell, Montgomery, and Pokrovskiy [4]. Sorting networks, as previously used in the work of Hyde et al. [16], could be used for the construction as well. However, this latter construction is more complicated than we need for our purposes here, and costs additional logarithmic factors.

Definition 4.1 (Hamilton router)**.** Suppose A and B are disjoint subsets of $V(S)$ with $|A| = |B|$ for some graph S. We call S an *A, B-Hamilton-router* if, for every bijection $\phi : A \to B$, $V(S)$ can be partitioned into a collection of paths, $\mathcal{P}_\phi$, with endpoints in $A \times B$ so that, letting M_ϕ be the matching $\{(\phi(a), a) \colon a \in A\} \subseteq B \times A$, the edge-set $E(\mathcal{P}_\phi) \cup M_\phi$ forms a Hamilton cycle over $V(S)$. The *order* of an A, B-Hamilton-router is $|A| = |B|$ and the *depth* of an A, B-Hamilton-router is $|V(S)|/|A| - 1$ (i.e., the average length of a component path in any path partition $\mathcal{P}_\phi$).

Remark. Throughout the paper, we will refer to Hamilton routers of order 2 as *comparators*.

The following proposition, whose proof is immediate by definition, explains why Hamilton routers are useful.

Proposition 4.2. *Let D be a directed graph containing an A, B-Hamilton-router S. Suppose that there exists a directed path forest $\mathcal{P}$ with exactly $|A| = |B|$ paths, start-vertices in B, end-vertices in A, and internal vertices partitioning $V(D) \setminus V(S)$. Then, D has a directed Hamilton cycle.*

4.1 Constructing Hamilton routers

We can build Hamilton routers of order n by gluing together Hamilton routers of order 2, i.e., comparators (as depicted in Figure 1), as was done by Bowtell et al. [4].

Definition 4.3. We call an A, B-Hamilton-router S a *basic Hamilton router of order n* if S can be constructed as follows. Start with a vertex set V defined on a $4 \times n$ grid, that is, $V := \{v_{i,j} : (i, j) \in [4] \times [n]\}$. Let $A = \{v_{1,j} : j \in [n]\}$ and $B = \{v_{4,j} : j \in [n]\}$.

- For every odd $i \in [n-1]$, and for $A' = \{v_{1,i}, v_{1,i+1}\}$ and $B' = \{v_{2,i}, v_{2,i+1}\}$, add a graph S_i^{odd} which is an A', B'-comparator.
- For every even $i \in [n-1]$, and for $A' = \{v_{3,i}, v_{3,i+1}\}$ and $B' = \{v_{4,i}, v_{4,i+1}\}$, add a graph S_i^{even} which is an A', B'-comparator.
- For every $i \in [n]$, add a path from $v_{2,i}$ to $v_{3,i}$.
- Add an edge from $v_{3,1}$ to $v_{4,1}$. If n is even, add an edge from $v_{3,n}$ to $v_{4,n}$, and otherwise add an edge from $v_{1,n}$ to $v_{2,n}$.

The short argument justifying why the above construction is a valid Hamilton router is given in [4].

Proposition 4.4 ([4])**.** *Basic Hamilton routers are well defined, that is, every (directed) graph constructed as in Definition 4.3 is an A, B-Hamilton-router.*

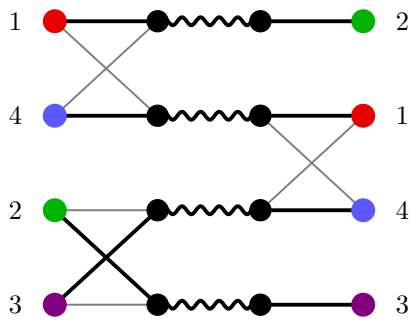

Figure 1: The above figure and following caption is borrowed from [4]. A basic Hamilton router of order 4 where (for simplicity) the comparators are depicted as 4-cycles. A and B are the leftmost and rightmost columns of vertices, respectively. An example bijection between A and B is indicated by the colour/number of the vertices. Choices of path systems for each comparator are indicated with thick lines, chosen so that identifying the vertex pairs of the same colour/number forms a cycle in the thick lines.

4.2 Building comparators out of even cycles

We want to embed Hamilton routers in mild expanders, so in particular, we need to be able to find comparators in such expanders. Expanders can have arbitrarily large girth, so C_4 is not a valid candidate for a comparator. Below we give a construction of comparators that can be built out of a single even cycle, followed by adding internally vertex-disjoint paths connecting various vertices of this base cycle. This eventually lets us find comparators with about $\log(n)^2$ vertices in expanders (after passing through Lemma 3.1). The idea is basically borrowed from [16], with the difference being that there is no modular restriction on the length of the even cycle. The construction we give here is also arguably simpler (but does not have the property of keeping the path lengths between the vertices the same, which is useful whilst embedding spanning trees).

Lemma 4.5. *For every even $k \geq 4$, there exists a graph G_k which is an A, B-Hamilton-router, where $A, B \subseteq V(G_k)$ are disjoint sets with $|A| = |B| = 2$, i.e. G_k is a comparator (recall Remark 1), and furthermore, G_k has the following properties.*

1. *$\Delta(G_k) \leq 3$*
2. *G_k can be obtained by starting with a k-cycle C, identifying some $A, B \subseteq V(C)$, appropriately pairing up the vertices of $V(C) \setminus (A \cup B)$, and, for each pair, adding a path of arbitrary length between its two vertices, so that the added paths are pairwise vertex-disjoint and each meets C precisely in its two endpoints.*

Proof. Property 2 in the lemma statement essentially gives away the construction of the comparator, save for how to identify A and B along the base cycle, and how to pair up the vertices of $V(C) \setminus (A \cup B)$. For both, there are plenty of choices that work, here we give an arbitrary one.

Let C be a k-cycle. If $k = 4$, there is nothing to do, as this already forms a comparator. Otherwise, enumerate the vertices as $c_0c_1c_2c_3c_4 \cdots c_{k-1}c_0$. Let $A = \{c_0, c_2\}$ and $B = \{c_1, c_4\}$. Let M_0 be the perfect matching of C matching c_0 to c_2, and let M_1 be the other perfect matching (which pairs c_1 with c_2).

Claim 2. There exists a perfect matching P of $V(C) \setminus (A \cup B)$ so that $M_0 \cup P$ and $M_1 \cup P$ are both acyclic.

Assuming the claim, P gives us the necessary pairing, and we argue that any G_k obtained by subdividing the edges of P (an arbitrary number of times) forms a valid A, B-Hamilton-router. Indeed, $M_0 \cup P$ and $M_1 \cup P$ are both spanning subgraphs where each vertex has degree 1 or 2 so both subgraphs are a union of cycles and paths, and there cannot be cycles by the claim, so just paths. The vertices in $A \cup B$ are the only vertices of degree 1, so these are the endpoints of the paths. As $M_0 \cup P$ has a path connecting c_0 and c_1, and $M_1 \cup P$ has a path connecting c_1 and c_2, it is clear that in each case, the remaining path is between the correct pair of vertices ($c_2 - c_4$ and $c_0 - c_4$, respectively), verifying the definition of a A, B-Hamilton-router.

The claim itself can be verified by giving an explicit pairing. For example,

$$P = \{c_{2j+1}c_{2j+4} : 1 \leq j \leq r-3\} \cup \{c_{2r-3}c_{2r-1}\}$$

is a pairing that does not create any cycles when combined with M_0 or M_1, see Figure 2. □

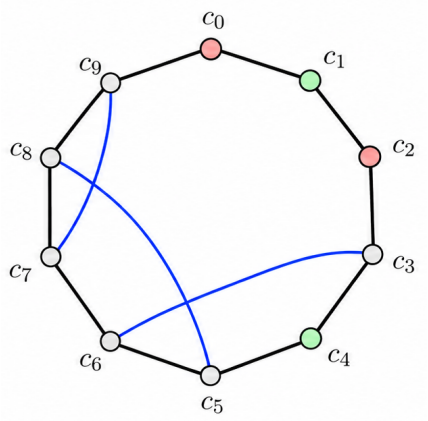

Figure 2: The graph G_k constructed in Lemma 4.5 when $k = 10$. The blue edges indicate the pairing given by P. Any graph in which the blue edges are subdivided, i.e. replaced by paths of arbitrary length with the same endpoints, forms a valid comparator.

4.3 Finding even cycles

To use the previous comparator construction, we need to be able to find many even length cycles in a given graph. We do not need expansion for this, just a degree bound. In particular, the following lemma can be proved easily by iterating the simple Moore bound, bounding the number of vertices of a graph in terms of its average degree and its girth, as given by a breadth-first search argument, see [1, Theorem 1].

Lemma 4.6 (Packing short even cycles)**.** *Let G be an n-vertex graph with $D \le \delta(G)$ and $\Delta(G) \le 2D$ for some $D \ge 100$. Then, G contains at least $n/(100 \log n)$ vertex-disjoint even-length cycles each of length at most $3 \log n$.*

Proof. Let H be a spanning bipartite subgraph of G obtained from a maximum cut. Then $e(H) \ge e(G)/2 \ge Dn/4$ and $\Delta(H) \le 2D$. Let $\mathcal{C}$ be a maximal collection of vertex-disjoint cycles in H, each of length at most $3 \log n$, and suppose that $|\mathcal{C}| < n/(100 \log n)$. If X is the union of these cycles, then $|X| < 3n/100$, and hence

$$e(H - X) \ge e(H) - 2D|X| > \frac{Dn}{4} - \frac{6Dn}{100} = \frac{19Dn}{100}.$$

Thus $H - X$ has average degree greater than $19D/50 \ge 38$. By the Moore bound for irregular graphs [1, Theorem 1], an N-vertex graph of average degree at least 38 contains a cycle of length at most $2\left\lceil \frac{\log(N/2)}{\log 37} \right\rceil + 2 < 3 \log n$. Since $H - X$ is bipartite, this cycle is even, contradicting the maximality of $\mathcal{C}$. Hence $|\mathcal{C}| \ge n/(100 \log n)$. □

4.4 Finding Hamilton routers in mild expanders

We can now combine our tools to give an embedding of a Hamilton router inside a random reservoir within an expander graph.

Lemma 4.7 (Random subsets of expanders contain Hamilton routers)**.** *There exist absolute constants $c > 0$, $K > 1$ and n_0 such that the following holds. Let H be an (n, d, λ)-graph with $n \ge n_0$ and $\lambda \le (1 - \delta)d$ for some $0 < \delta \le 1/2$. Let $0 < q \le \delta/100$ with $qn \in 2\mathbb{N}$, and suppose that $qd \ge K\delta^{-4}(\log n)^2$. Let $R \subseteq V(H)$ be chosen among the qn-element subsets of $V(H)$ uniformly at random. Then, with probability at least $1 - 1/n$, the following holds simultaneously for all disjoint sets $A, B \subseteq V(H) \setminus R$ with $|A| = |B| = k$.*

If every $x \in A \cup B$ satisfies $d_H(x, R) \ge \frac{qd}{2}$, every $v \in R$ satisfies $d_H(v, A \cup B) \le c\frac{\delta^3 qd}{\log n}$, and it holds that $k \le c\delta^3|R|(\log n)^{-2}$, then $H[A \cup B \cup R]$ contains an A, B-Hamilton router S such that $A \cup B \subseteq V(S) \subseteq A \cup B \cup R$.

Proof. Let $c_0 > 0$ be a sufficiently small absolute constant, set $s := \lfloor c_0\delta^2 qn/\log n \rfloor$, and set $m := (qn-s)/2$. We may generate a uniform qn-set R by first choosing disjoint sets C, U, V uniformly with $|C| = s$ and $|U| = |V| = m$, and then setting $R = C \cup U \cup V$. Write $D_0 := md/n$ and $D_C := sd/n$. For K sufficiently large, $D_0 \geq qd/3$ and $D_C \geq c_0\delta^2 qd/(2\log n) \geq C\log n$.

Apply Lemma 2.2 to (U, V) with $a = 4$. Standard hypergeometric Chernoff bounds, followed by a union bound over all vertices, also give simultaneously $d_H(v, U), d_H(v, V) = (1 \pm \delta/100)D_0$ and $d_H(v, C) = (1 \pm 1/10)D_C$ for every $v \in V(H)$. Thus, with probability $1 - o(n^{-1})$,

$$s_2(H[U,V]) \leq (1-\delta/2)D_0, \qquad d_{H[U,V]}(x) = (1 \pm \delta/100)D_0, \qquad d_H(v, C) \leq 2D_C$$

for all relevant vertices. We henceforth fix such a partition.

Let $J := H[U, V]$, viewed as a bipartite graph on Q. Since the second largest adjacency eigenvalue of J is $s_2(H[U, V])$, Lemma 2.1 gives that J is ρ-cut-dense for $\rho := \delta d/(5n)$. Moreover, $\Delta(J) \leq (1 + \delta/100)D_0$, and therefore $\rho|Q| \geq \frac{\delta}{3}\Delta(J)$. In particular, Lemma 3.1 applies to J with $\gamma = \delta/3$ provided the degree conditions hold.

Next, we embed the base cycles of the comparators. The graph $H[C]$ has minimum degree at least $D_C/2$ and maximum degree at most $2D_C$. By Lemma 4.6, $H[C]$ contains at least $c_1|C|\log(n)^{-1}$ vertex-disjoint even cycles, each of length at most $g := C_1 \log n$. Since $k \leq c\delta^3 qn \log^{-2} n$ and $|C| = \Theta(\delta^2 qn/\log n)$, choosing c sufficiently small ensures that there are at least k such cycles. A basic Hamilton router of order k uses fewer than k comparators, so we can assign a distinct cycle to each comparator and apply Lemma 4.5 to designate its four input/output vertices (A, B) and pair its remaining vertices.

It remains to form all the required connections between the cycles, both to complete the even cycles into comparators and to join the comparators following the template of a basic Hamilton router. Form a collection $\mathcal{D}$ of pairs whose endpoints lie in $A \cup B \cup C$ as follows. Include all pairs used in the comparator constructions, and include one pair for every fixed connection joining the inputs/outputs of the comparators in the definition of a basic Hamilton router. Every vertex occurs only a bounded number of times in $\mathcal{D}$ (in fact the construction gives much less than 10). Every endpoint x of a pair in $\mathcal{D}$ has at least $qd/3$ neighbours in Q. Indeed, for $x \in A \cup B$ this follows from $d_H(x, R) \geq qd/2$ and $d_H(x, C) \leq 2D_C$, and for $x \in C$ it follows directly from the degree concentration into U and V.

Let T be the set of vertices occurring in $\mathcal{D}$. Since all cycle inputs/outputs lie in C, for every $v \in Q$ we have $d_H(v, T) \leq d_H(v, C) + d_H(v, A \cup B) \leq 2D_C + c\delta^3 qd/\log n$. By the choice of s, this is at most $c_2\delta^2 qd/\log n$, where c_2 can be made arbitrarily small by choosing c_0 and then c sufficiently small. As $\Delta(J) = \Theta(qd)$, this is precisely the condition required in Lemma 3.1 with β an absolute constant and $\gamma = \Theta(\delta)$. The same lemma therefore provides internally vertex-disjoint paths through Q joining every pair in $\mathcal{D}$.

Combining these paths with the base cycles gives the required comparators by Lemma 4.5, and the remaining connecting paths give the fixed connections of a basic Hamilton router. Replacing those connections by subdivisions does not affect the proof of Proposition 4.4, so their union is an A, B-Hamilton router S with $A \cup B \subseteq V(S) \subseteq A \cup B \cup R$. □

5 Checking Hall's condition between random sets after perturbation

Below we give a convenient criterion for when a balanced bipartite graph contains a perfect matching. In the below, H_0 is a balanced bipartite graph, which, in our application, will be obtained by random sampling from a regular expander. We will show such an H_0 is nearly-regular and has cut-density (using Chernoff and Lemma 2.2), as assumed in the below statement. The cut-density parameter exceeds the error from near-regularity, so Hall's condition holds for H_0 with some slack. This slack is sufficient to say that H_0 still has a perfect matching even if a few more vertices are added each of which have large degree towards H_0, provided that each vertex of H_0 sends not too many edges towards the new vertices.

The proof is a simple and standard counting argument, we give the details for the sake of completeness.

Lemma 5.1. *Let H_0 be a balanced bipartite graph with bipartition (L_0, R_0), where $|L_0| = |R_0| = m$. Suppose that H_0 is p-cut-dense and $d_{H_0}(v) = (1 \pm \eta)D$ for every $v \in V(H_0)$. Set $\alpha := pm/D$, and suppose that $0 < \alpha \leq 1/2$ and $\eta \leq \alpha/10$.*

Let X, Y be disjoint sets with $|X| = |Y|$, and let H be a bipartite graph with bipartition $(L_0 \cup X, R_0 \cup Y)$ containing H_0. Suppose that every $x \in X$ has at least $D/2$ neighbours in R_0, every $y \in Y$ has at least $D/2$ neighbours in L_0, and

$$d_H(v, X) \le \frac{\alpha D}{20} \quad \text{for every } v \in R_0, \qquad d_H(v, Y) \le \frac{\alpha D}{20} \quad \text{for every } v \in L_0.$$

Then H contains a perfect matching.

Proof. If $S \subseteq L_0$ and $T \subseteq R_0$ satisfy $e_{H_0}(S, T) = 0$, then

$$m - |S| - |T| \ge \alpha \min\{|S|, |T|\}.$$

Indeed, write $s := |S|$, $t := |T|$, $h := m - s - t$, and assume $s \le t$. For $Z := S \cup (R_0 \setminus T)$ we have $|Z| = 2s + h \le m$. Hence cut-density and near-regularity give $pm(2s+h) \le e_{H_0}(Z, V(H_0) \setminus Z) \le D(2\eta s + (1+\eta)h)$. Since $pm = \alpha D$ and $\eta \le \alpha/10$, this implies $h \ge \alpha s$, proving the claim.

Suppose now that H has no perfect matching. By Hall's theorem there are sets $S' \subseteq L_0 \cup X$ and $T' \subseteq R_0 \cup Y$ with $e_H(S', T') = 0$ and $|S'| + |T'| > m + |X|$. Write $S = S' \cap L_0$, $T = T' \cap R_0$, $x := |S' \cap X|$, $y := |T' \cap Y|$, and $h := m - |S| - |T|$. Since $x, y \le |X|$, Hall's condition failing gives $h < \min\{x, y\}$.

Every vertex of $S' \cap X$ sends at least $D/2$ edges into $R_0 \setminus T$, while every vertex there receives at most $\alpha D/20$ such edges. Thus $x \le (\alpha/10)(m - |T|)$, and similarly $y \le (\alpha/10)(m - |S|)$. Consequently, writing $u := \min\{|S|, |T|\}$, $h < \frac{\alpha}{10}(h + u)$. On the other hand, the claim above gives $h \ge \alpha u$. Hence $h < \frac{\alpha+1}{10} h \le \frac{3}{20} h$, a contradiction. □

6 Putting everything together

Proof of Theorem 1.3. We first describe the proof at a high level. We first choose a small random set R, and partition the remainder randomly into equal-sized sets. We will sometimes call R a *reservoir* and the latter random sets *layers*. We build a Hamilton router with terminal sets A, B inside $A \cup B \cup R$, where A and B are layers, then redistribute every vertex left outside the router among the remaining layers and use Lemma 5.1 to find perfect matchings between consecutive layers. The resulting k disjoint B–A paths from the union of perfect matchings cover everything outside the router, so Proposition 4.2 finishes the proof.

We suppress divisibility issues throughout to ease exposition. In particular, we assume that all set sizes below are integers and that k divides $n - |V(S)|$ once the router S has been found. These assumptions can be removed by starting the proof by finding a short path whose remainder is appropriately divisible, contracting the vertices of the short path, and essentially following the below proof verbatim.

Choose sufficiently small absolute constants $0 < c_1 \ll c_0 \ll 1$, and put $q := c_0\delta$ and $k := c_1\delta^3 qn/L^2$. We may assume $0 < \delta \le 1/2$, since the case of larger δ is only easier. Set $h := 2qk$ and $m := k - h = (1 - 2q)k$, and let $T := \lfloor (n - qn - 2k)/m \rfloor$. Note that $T = \Theta(n/k)$ and, since $qn \gg k$, we have $T \ge n/k$ for n sufficiently large.

Choose uniformly at random a partition

$$V(G) = R \sqcup A_0 \sqcup A_1 \sqcup B_0 \sqcup B_1 \sqcup C_1 \sqcup \cdots \sqcup C_T \sqcup Z,$$

where $|R| = qn$, $|A_0| = |B_0| = |C_i| = m$ and $|A_1| = |B_1| = h$. Set $A := A_0 \cup A_1$ and $B := B_0 \cup B_1$, so $|A| = |B| = k$.

Part 1. We first record several properties of this partition that each hold with high probability.

We start with the spectral properties of the various induced bipartite subgraphs. Set $D := md/n$. Apply Lemma 2.2 (with $p \to m/n$ and the same value of D, δ), with a sufficiently large fixed value of a, to every pair (C_i, C_{i+1}), to (B_0, C_1), and to (C_i, A_0) for every $i \le T$. Each such pair is genuinely a uniformly random ordered pair of disjoint m-sets of the original graph, and the hypotheses $p \le \delta/100$ and $D \ge C_a\delta^{-2}\log n$ hold because $m = (1 - 2q)k$, $q = c_0\delta$, and $k = c_1\delta^3 qn/(\log n)^2$, so, for c_0, c_1 sufficiently small and n sufficiently large,

$$\frac{m}{n} \le \frac{k}{n} = c_0 c_1 \frac{\delta^4}{(\log n)^2} \le \frac{\delta}{100}, \qquad D \ge \frac{c_0 c_1}{2} \frac{\delta^4 d}{(\log n)^2} \ge C_a \delta^{-2} \log n,$$

where the last inequality follows from $d \ge C\delta^{-6}(\log n)^3$ once the absolute constant C is chosen sufficiently large.

The conclusion of Lemma 2.2 ensures that in each application, the corresponding random layers form a bipartite graph with good spectral gap, i.e. the corresponding biadjacency matrix K has $s_2(K) \leq (1-\delta/2)D$ (meaning the adjacency matrix A of the bipartite graph has second largest eigenvalue at most $(1-\delta/2)D$), with probability $1-n^{-a}$. As $T = O(\delta^{-4}\log(n)^2)$, we may take a union bound to conclude with high probability, this conclusion holds for all pairs of random sets.

We now record the concentration of degrees. Set $\eta := \delta/100$. Standard hypergeometric concentration gives that with high probability $d_G(v,X) = (1\pm\eta)D$ holds for every vertex v and every layer $X \in A_0, B_0, C_1, \dots, C_T$. We can similarly assert, with high probability, the corresponding degree concentration of each vertex into $\{R, A_0, A_1, B_0, B_1\}$ and any other random set that is obtained by unions and complements from this set, such as A and B.

Consequently, for every relevant pair of random layers, the balanced bipartite graph between them has all degrees $(1\pm\eta)D$ and second adjacency eigenvalue at most $(1-\delta/2)D$. Lemma 2.1 therefore shows that each such pair is p_*-cut-dense, where we may safely take $p_* := \frac{\delta d}{10n}$.

Finally, by Lemma 4.7, with probability at least $1-1/n$ the random set R has the required Hamilton router property simultaneously for all admissible A, B.

Part 2. By the probabilistic method, we may fix a partition that satisfies all of the properties above. The rest of the proof uses these properties deterministically to conclude the existence of a Hamilton cycle.

Let us start by considering the reservoir R, in which we wish to embed a Hamilton router. By the degree bounds, the sets A and B satisfy the hypotheses from Lemma 4.7. Indeed, every $x \in A \cup B$ has $d_G(x,R) \geq qd/2$, while every $v \in R$ has $d_G(v, A\cup B) \leq 4kd/n \leq c\delta^3 qd/L$, provided c_1 was chosen sufficiently small. Since $k \leq c\delta^3|R|\log\log n/\log(n)^2$, the property from Lemma 4.7 gives an A,B-Hamilton router $S \subseteq G[A\cup B\cup R]$.

Next, we wish to partition the vertices outside the router S, and combine them with the layers $C_1, \dots, C_T$, to form various balanced bipartite graphs between which we intend to find perfect matchings. We will end up discarding a few layers C_i for convenience, and redistribute. Set $t := (n-|V(S)|)/k$. Since $S \subseteq A\cup B\cup R$, we have $t \leq n/k \leq T$. Let $P := V(G) \setminus \big(V(S)\cup C_1 \cup \dots \cup C_t\big)$. By the divisibility hypothesis and the definition of m, $|P| = t(k-m) = th$. Partition P uniformly at random into sets $X_1, \dots, X_t$ of size h, and set $L_i := C_i \cup X_i$. Thus every L_i has size k and $V(G)\setminus V(S) = L_1 \sqcup \dots \sqcup L_t$.

The X_is are random sets at the moment, and we wish to show that with high probability, they are 'admissible' with respect to the deterministic sets C_i from the perspective of the hypotheses of Lemma 5.1. Set $\alpha := \frac{p_* m}{D} = \frac{\delta}{10}$, noting $\eta \leq \alpha/10$, exactly as required in Lemma 5.1. Our initial concentration event gives $d_G(v,P) = O(qd)$ for every v. The sets X_i form a uniform equipartition, so another hypergeometric concentration estimate gives, simultaneously for all v and i, $d_G(v,X_i) \leq 10qD$, with high probability. Since $q = c_0\delta$ and c_0 is sufficiently small, this is at most $\alpha D/20$. Similarly, the earlier concentration bounds give $d_G(v,A_1), d_G(v,B_1) \leq \alpha D/20$ for every vertex. On the other hand, every vertex of every X_i, and every vertex of $A_1\cup B_1$, has at least $(1-\eta)D \geq D/2$ neighbours in each layer C_i.

By the probabilistic method, we may fix the X_i to have the degree conditions required in the above paragraph. We can therefore apply Lemma 5.1 successively, first with $H_0 \to G[B_0, C_1]$ and $X, Y \to B_1, X_1$, and later applications with $H_0 \to G[C_i, C_{i+1}]$ and $X, Y \to X_i, X_{i+1}$, and a final time with $H_0 \to G[C_t, A_0]$ and $X, Y \to X_t, A_1$. In every case the hypotheses of Lemma 5.1 hold, so there are perfect matchings between B and L_1, between L_i and L_{i+1} for every $i < t$, and between L_t and A.

The union of these matchings is a collection $\mathcal{P}$ of exactly k vertex-disjoint paths, each starting in B and ending in A. Since every intermediate matching is perfect, the internal vertices of these paths are exactly $L_1 \cup \dots \cup L_t = V(G)\setminus V(S)$. Thus $\mathcal{P}$ is precisely the spanning path forest required by Proposition 4.2. Combining $\mathcal{P}$ with the A,B-Hamilton router S gives a Hamilton cycle in G. □

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